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# On the Algebraic Structure of Gravitational Descendants in $CP(n-1)$ Coset Models

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## Abstract

We investigate how specific free-field realizations of twisted  $N=2$  supersymmetric coset models give rise to natural extensions of the “matter” Hilbert spaces in such a manner as to incorporate the various gravitational excitations. In particular, we show that adopting a particular screening prescription is equivalent to imposing the requisite equivariance condition on cohomology. We find a simple algebraic characterization of the  $W_n$ -gravitational ground ring spectra of these theories in terms of affine- $SU(n)$  highest weights.

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## 1. Introduction

From the work of [1,2] it is evident that there should be some beautiful and simple algebraic structure underlying the coupling of topological  $W$ -matter theories to topological  $W$ -gravity [3]. Our purpose in this paper is to simplify and then generalize the results of [2] and thereby elucidate the algebraic structure.

The key idea accomplishing this is to find a matter representation of the gravitational descendants. Specifically, this means expanding the unitary matter theory in a natural manner to a (necessarily) non-unitary theory, and then finding representations of the gravitational descendants within the extended matter Hilbert space. The basic idea was first suggested in [4] for topological Landau-Ginzburg theories coupled to ordinary topological gravity (it also surfaced in the work of [1]). This idea was then developed and greatly refined in [5], where the authors introduced a computationally (and conceptually) invaluable simplification in the form of a global Hilbert space rotation that maps the  $BRST$  operator into a much simpler expression.

The problem in applying these techniques to more general models is to find a systematic and natural extension of the matter Hilbert space so as to incorporate the various kinds of gravitational descendants. For our problem, where we consider topological  $W$ -minimal models [6,7] coupled to  $W$ -gravity, the natural Hilbert space extension is obtained by removing the fermionic screening prescription in the Drinfeld-Sokolov construction of these models. This particular prescription was suggested in [1], which considered a specific Drinfeld-Sokolov reduction that yields a non-standard free-field realization of these models. Though this specific free-field realization was natural in the context [8] of the problems discussed in [1], it was prohibitively inconvenient for discussing the full ground ring [9] spectrum. In the Landau-Ginzburg formulation of [2], on the other hand, the full spectrum is relatively easy to obtain, but its algebraic structure remained obscure.

In the present paper we will study the modified fermionic screening prescription in the standard Drinfeld-Sokolov type free-field realization, and show that it leads to a very simple description of the  $W_n$ -gravitational descendants directly in terms of weight lattices of  $SU(n)$ . Specifically, in the next section we will briefly review the results of [5,2], while in section 3, after summarizing the relevant parts of the  $N=2$  supersymmetric free-field realization, we will construct the gravitational excitations in terms of free-field vertex operators and find a simple algebraic characterization of the complete spectrum.

## 2. Equivariant cohomology in topological matter-gravity systems

We will define topological  $W_n$ -gravity coupled to  $W_n$ -matter rather formally by thinking of it as a multi-field generalization of the theories introduced in [10–12]. That is, the Liouville, matter and ghost sectors are separately topologically twisted  $N=2$  superconformal systems, each consisting of  $2n-2$  superfields [2]. We will denote the corresponding currents of the topological algebra by  $J_{(i)}, G_{(i)}^+, G_{(i)}^-, T_{(i)}$ , where  $i = l, gh, m$ <sup>†</sup>. There are two anti-commuting *BRST* charges,

$$\begin{aligned} Q_s &= \frac{1}{2\pi i} \oint dz G_{tot}^+ \equiv \frac{1}{2\pi i} \oint dz (G_{(l)}^+ + G_{(m)}^+ + G_{(gh)}^+) \\ Q_v &= \frac{1}{2\pi i} \oint dz d\theta^+ \left( \sum_{j=1}^{n-1} C_{-j} \widetilde{W}_{j+1} \right), \end{aligned} \quad (1)$$

where  $C_{-j}$  is the  $j$ -th ghost superfield with spin  $-j$  and  $\widetilde{W}_{j+1} \equiv D_-(W_{j(l)} + W_{j(m)} + \frac{1}{2}W_{j(gh)})$ , and where  $W_{j(i)}$  is the spin  $j$  current superfield in the sector  $(i)$ . Note that due to the non-linearity of the  $W$ -algebra,  $W_{j(gh)}$  is very complicated for  $j > 1$  [2,13], and depends upon both the matter and Liouville fields.

Following [5,2] we introduce the operator

$$S = \exp \left[ \frac{1}{2\pi i} \oint dz \left( \sum_{j=1}^{n-1} c_{-j} \widetilde{U}_{j+1} \right) \right], \quad (2)$$

where  $c_{-j} \equiv C_{-j}|_{\theta=0}$  and  $\widetilde{U}_{j+1} \equiv \widetilde{W}_{j+1}|_{\theta=0}$ . One can then show that

$$\begin{aligned} S^{-1} Q_{tot} S &\equiv S^{-1} (Q_s + Q_v) S = Q_s \\ S^{-1} b_{j+1} S &= b_{j+1} + U_{j+1}. \end{aligned} \quad (3)$$

The cohomology problem is thus reduced to the computation of the cohomology of  $Q_s$ . Moreover, the condition for equivariant cohomology:  $b_{j+1,0} |\text{state}\rangle = 0$ , which yields the operators of topological  $W_n$ -gravity, is changed to

$$(b_{j+1} + U_{j+1})_0 |\text{state}\rangle = 0, \quad j = 1, \dots, n-1. \quad (4)$$

At this point it should be stressed that, strictly speaking, the foregoing has been explicitly verified only for  $n = 2$  [5] and  $n = 3$  [2]. However, we think it is highly

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<sup>†</sup> These subscripts refer to Liouville, ghost and matter sectors respectively.

plausible that it is true in general. Thus, we will assume in the following that the physical spectrum of topological  $W_n$ -gravity coupled to  $W_n$ -matter can be obtained for general  $n$  from the cohomology of  $Q_s$  modulo the equivariance conditions (4).

The problem of finding matter representatives of all physical states is now considerably simplified since  $Q_s$ , and the equivariance condition, do not mix the matter, Liouville and ghost Hilbert spaces. Thus a pure matter physical state,  $|X\rangle$ , must satisfy

$$\begin{aligned} Q_{s,(m)} |X\rangle &= 0 \\ (U_{j+1,(m)})_0 |X\rangle &= 0, \quad j = 1, \dots, n-1, \end{aligned} \tag{5}$$

and it is trivial precisely if there is some  $|Y\rangle$  such that:

$$\begin{aligned} |X\rangle &= Q_{s,(m)} |Y\rangle, \quad \text{with} \\ (U_{j+1,(m)})_0 |Y\rangle &= 0, \quad j = 1, \dots, n-1. \end{aligned} \tag{6}$$

A further simplification is obtained if we work with states that are simultaneously highest weight states of the  $N=2$  superconformal algebra and eigenstates of the zero modes of the super- $W_n$  currents. Then the equivariance conditions reduce to  $(G_{(m)}^-)_0 |\text{eigenstate}\rangle = 0$ , since the other conditions follow from the commutation relations of the super- $W_n$  algebra.

The physical operators of topological  $W$ -gravity are the gravitational descendants,  $\gamma_{0,p}$ ,  $p = 1, \dots, n-1$ , and polynomials thereof [3]. They are *BRST*-exact,  $\gamma_{0,p} = \{Q_s, \partial^p c_{-p}\}$ , but since  $\partial^p c_{-p}$  do not satisfy the equivariance conditions, the  $\gamma_{0,p}$  are non-trivial operators. To find corresponding representatives in the matter sector,  $\Gamma_{0,p}(z)$ , we must find operators  $\Lambda_p(z)$  such that  $\Gamma_{0,p}(z) = \{Q_s, \Lambda_p(z)\}$  and such that  $(\partial^p c_{-p}(z) - \Lambda_p(z))$  satisfies the equivariance conditions. That is, we must have

$$(U_{j+1,(m)})_0 |\Lambda_p\rangle = a_j \delta_{jp}, \tag{7}$$

for some non-zero constants  $a_j$ . In particular, this means that in terms of the twisted  $N=2$  superconformal algebra, we have

$$\Gamma_{0,1} = G_{-1/2}^+ \Lambda_1, \quad \text{but} \quad G_{1/2}^- \Lambda_1 = \text{const}.$$

This directly violates the Hodge decomposition theorem [14] for unitary  $N=2$  superconformal theories, and therefore establishes that a matter representation of gravitational descendants can only be found in a non-unitary extension of the matter system.

For the particular representation of these theories (with  $n = 2, 3$ ) in terms of Landau-Ginzburg models, it was shown in [5,2] that arbitrary polynomials in the Landau-Ginzburg variables provides a natural extension of the matter Hilbert space, and yields the requisite matter representation of the gravitational descendants.

### 3. Free-field realization

To illustrate our ideas we will consider the  $N=2$   $W_3$  minimal models, which are based on cosets  $\frac{SU(3)_k}{U(2)}$  [6]. They can be constructed via Drinfeld-Sokolov reduction from  $SU(3|2)$  in terms of free superfields [7]; in particular, the super- $W_3$  generators were explicitly given in [7,15]. Here we will give simply the (left-moving)  $N=2$  superconformal generators in component fields:

$$\begin{aligned}
J(z) &= [\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2 - i \alpha_0 (\partial \phi_1 + \partial \phi_2 - \partial \bar{\phi}_1 - 2 \partial \bar{\phi}_2)](z) \\
G^+(z) &= \sqrt{2} [\bar{\psi}_1 \partial \phi_1 + \bar{\psi}_2 \partial \phi_2 + i \alpha_0 (\partial \bar{\psi}_1 + 2 \partial \bar{\psi}_2)](z) \\
G^-(z) &= \sqrt{2} [\psi_1 \partial \bar{\phi}_1 + \psi_2 \partial \bar{\phi}_2 + i \alpha_0 (\partial \psi_1 + \partial \psi_2)](z) \\
T(z) &= \sum_{j=1}^2 \left[ -\partial \phi_j \partial \bar{\phi}_j + \frac{1}{2} (\psi_j \partial \bar{\psi}_j + \bar{\psi}_j \partial \psi_j) \right](z) \\
&\quad - \frac{i}{2} \alpha_0 (\partial^2 \phi_1 + \partial^2 \phi_2 + \partial^2 \bar{\phi}_1 + 2 \partial^2 \bar{\phi}_2)(z)
\end{aligned} \tag{8}$$

where  $\alpha_0 = 1/\sqrt{k+3}$  and where we have used the conventions:  $\phi_i(z) \bar{\phi}_j(w) \sim -\delta_{ij} \log(z-w)$ ,  $\psi_i(z) \bar{\psi}_j(w) \sim -\delta_{ij} (z-w)^{-1}$ . The Drinfeld-Sokolov reduction also provides the screening operators that are needed to truncate the free-field spectrum. These screening operators fall into two classes, the  $D$ -type screeners,

$$\begin{aligned}
R_1^D &= [(\partial \phi_1 - \partial \phi_2 + \partial \bar{\phi}_2) + 2i \alpha_0 (\psi_1 - \psi_2) \bar{\psi}_2] e^{-i \alpha_0 (\phi_2 - \phi_1 + \bar{\phi}_2)} \\
R_2^D &= [(\partial \phi_1 - \partial \bar{\phi}_1) + 2i \alpha_0 \psi_1 \bar{\psi}_1] e^{-i \alpha_0 (\phi_1 + \bar{\phi}_1)} \\
S_1^D &= [(\partial \phi_1 - \partial \phi_2 + \partial \bar{\phi}_1) - 2i \alpha_0 (\psi_1 - \psi_2) \bar{\psi}_1] e^{i \alpha_0 (\phi_2 - \phi_1 + \bar{\phi}_1)},
\end{aligned} \tag{9}$$

and the  $F$ -type, or fermionic, screeners:

$$\begin{aligned}
P_1^F &= i \psi_1 e^{\frac{i}{\alpha_0} \phi_1}; & P_3^F &= i \bar{\psi}_1 e^{\frac{i}{\alpha_0} \bar{\phi}_1} \\
P_2^F &= i (\psi_2 - \psi_1) e^{\frac{i}{\alpha_0} (\phi_2 - \phi_1)}; & P_4^F &= i \bar{\psi}_2 e^{\frac{i}{\alpha_0} \bar{\phi}_2}
\end{aligned} \tag{10}$$

The unitary, minimal superconformal model is obtained by using a  $BRST$  operator constructed from all of the  $D$ -type screeners, and half of the  $F$ -type screeners, for example  $P_1^F$  plus  $P_2^F$ . The combined topological matter-gravity systems can thus be described by combining this free-field  $BRST$  operator with the  $BRST$  operator  $Q_s + Q_v$  described in section 2.

The idea is now to perform the  $BRST$  reduction in a slightly different manner. Namely, first perform the  $D$ -type  $BRST$  reduction, and combine the  $F$ -type screening

charges obtained from  $P_1^F$  and  $P_2^F$  with the supercharge  $Q_s$  in (1). One can perform the  $F$ -type  $BRST$  reduction at any point, because the  $BRST$  charge is local with respect to the other screening currents, and anti-commutes with  $Q_s$  and the  $D$ -type screeners. (Note that one can view the left-moving  $F$ -type screeners as the left-moving part of the “right-moving” supercharge  $\tilde{G}^+(z)$  (and similarly for  $G^+(z)$ ), such that they behave exactly like the terms  $\sum \psi_i \frac{\partial W}{\partial x_i}$  of the supercurrents in the Landau-Ginzburg formulation [16].)

Before we discuss the reduction method in more detail, we think it valuable to say more precisely what is the underlying idea of performing the  $D$ -type reduction while not performing the  $F$ -type reduction. A simple way to understand this is to consider the  $N=2$  superconformal minimal models, whose coset form is  $\frac{SU(2)_k \times SO(2)}{U(1)}$ . If one uses the Kac-Wakimoto realization of  $SU(2)_k$ , one introduces a superghost system,  $(\beta, \gamma)$ , plus a single free boson. To form the coset model, one bosonizes the superghosts according to [17]

$$\begin{aligned}\beta &= (\partial\xi)e^{-\phi} = i(\partial\chi)e^{i\chi-\phi} \\ \gamma &= \eta e^{\phi} = e^{-i\chi+\phi}\end{aligned}\tag{11}$$

To recover the Hilbert space of the  $(\beta, \gamma)$  system one has to fix the momenta  $p_\phi - p_\chi$  and exclude all states involving the zero mode,  $\xi_0$ . Equivalently, one can allow states with  $p_\phi - p_\chi \geq 0$ , and then compute the cohomology of the fermionic charge  $Q = \oint \eta$  [18]. This is precisely what is accomplished by the fermionic screener  $P^F$  in the superconformal model. That is, if one does not employ the fermionic screener, one obtains infinitely many copies of the physical “matter” Hilbert space, related by spectral flow in  $p_\phi - p_\chi$ . From the following it will be clear that these copies can be reinterpreted as gravitational descendants of the matter sector.

According to what was said above, we take

$$Q_{s,(m)} = \frac{1}{2\pi i} \oint dz G^+(z) - i \frac{1}{2\pi i} \oint dz (P_1^F + P_2^F)(z) .\tag{12}$$

One easily finds that

$$\begin{aligned}e^{\frac{i}{\alpha_0}\phi_1} &= \{Q_{s,(m)}, (\bar{\psi}_1 + \bar{\psi}_2)\} \\ e^{\frac{i}{\alpha_0}(\phi_2 - \phi_1)} &= \{Q_{s,(m)}, \bar{\psi}_2\} .\end{aligned}\tag{13}$$

Moreover, one has

$$G_{1/2}^-\psi_j = i\sqrt{2}\alpha_0 = \text{const.}$$

This makes the vertex operators in (13) into candidates for matter representatives of the pure  $W$ -gravity descendants, and according to our considerations in section 2, we simply need to verify equation (7). This is most simply accomplished by showing that there are linear combinations of the  $\psi_i$  that are eigenstates (with distinct eigenvalues) of the zero mode,  $S_0$ , of the lowest, spin-2 component of the  $W_3$  supercurrent.

The easiest way to isolate this spin-2 current is to remember that the  $N=2$  super coset model factorizes according to:

$$\frac{SU(3)_k \times SO(4)}{SU(2)_{k+1} \times U(1)} \cong \frac{SU(3)_k}{SU(2)_k \times U(1)} \otimes \frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}} \otimes U(1) . \quad (14)$$

The second factor is a Virasoro minimal model, and its Coulomb gas realization is embedded in the  $N=2$  free-field formulation above [19,15]. Specifically, define a scalar field,  $\varphi$ , by

$$\partial\varphi = -\sqrt{\frac{k+3}{2(k+2)}}(\partial H_1 - \partial H_2) - \frac{1}{\sqrt{2(k+2)}}(\partial\phi_1 - \partial\phi_2 - \partial\bar{\phi}_1) , \quad (15)$$

where  $\psi_j = e^{iH_j}$ ,  $\bar{\psi}_j = -e^{-iH_j}$ . The stress tensor  $T(z)$  in (8) contains a part that is precisely

$$T^\varphi = -\frac{1}{2}(\partial\varphi)^2 - \frac{i}{\sqrt{2(k+2)(k+3)}}\partial^2\varphi ,$$

and this is the natural candidate for the stress tensor of the minimal model. There is, however, a subtlety. The operator  $T^\varphi$  does not quite commute with the screening charges of the  $N=2$  theory. The operator that does commute with the screening charges rather is:

$$\begin{aligned} \widehat{T}^\varphi(z) &= T^\varphi(z) + N(z) , \quad \text{where} \\ N(z) &= \frac{i}{(k+2)\sqrt{k+3}}(\psi_1\bar{\psi}_2)\partial\bar{\phi}_1 - \frac{1}{k+2}(\partial\psi_1)\bar{\psi}_2 . \end{aligned} \quad (16)$$

The origin and interpretation of this extra term will be described in [15]. For the present we simply observe that it modifies the computation of the eigenstates.

In terms of the decomposition (14), the spin-2 current is then given by

$$S(z) = T(z) - \frac{1}{4k(k+3)}J^2(z) - \frac{(k+2)(5k-3)}{k(k+5)}\widehat{T}^\varphi(z) . \quad (17)$$

This is a good conformal field, with normalization  $S(z)S(0) \sim \frac{(5k-3)(2k+3)(k-1)}{k(k+3)(k+5)} \frac{1}{z^4}$ . From (17) we find that  $\bar{\psi}_2$  and  $\bar{\psi}_1 + \bar{\psi}_2$  are eigenstates of  $S_0$ , with eigenvalues:  $-\frac{(k^2+3)}{k(k+5)}$

and  $-1$ , respectively. This proves that indeed certain linear combinations of the matter vertex operators (13) are *BRST* representatives of the gravitational excitations,  $\gamma_{0,1}, \gamma_{0,2}$ .

In the  $N = 2$  superconformal model, the elements of the primary chiral ring (which constitutes the “matter” sector) can be very simply represented by<sup>†</sup>

$$V_{a,b} = e^{\frac{i}{\sqrt{k+3}}[a\phi_1 + b(\phi_2 - \phi_1)]}, \quad (18)$$

for  $0 \leq b \leq a \leq k$ . Based on the fact that  $\gamma_{0,1}$  and  $\gamma_{0,2}$  can be represented by  $V_{k+3,0}$  and  $V_{0,k+3}$ , we expect that the gravitational descendants can be represented by the vertex operators  $V_{a,b}$  with  $a, b \geq 0$ . We will show that this is the case, but that one needs to make the further restriction:  $0 \leq b \leq a$  in order to get one copy of each of the gravitationally dressed physical states. (These “gravitational” ring elements will be chiral, but not primary conformal fields.) The problem is to determine which of these vertex operators correspond to physical states and which correspond to null states. There are two parts to this. First, we must consider the equivariant cohomology. Obviously:

$$\begin{aligned} V_{a,b} &= \{Q_{s,(m)}, (\bar{\psi}_1 + \bar{\psi}_2) V_{a-(k+3),b}\} \\ &= \{Q_{s,(m)}, \bar{\psi}_2 V_{a,b-(k+3)}\}, \end{aligned} \quad (19)$$

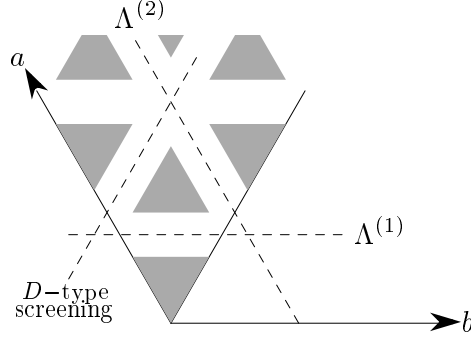
and one can verify that  $\Lambda_{a',b'}^{(1)} \equiv (\bar{\psi}_1 + \bar{\psi}_2) V_{a',b'}$  and  $\Lambda_{a',b'}^{(2)} \equiv \bar{\psi}_2 V_{a',b'}$  are eigenstates of  $S_0$ . It therefore suffices to check whether  $G_{1/2}^- \Lambda_{a',b'}^{(i)}$  are physical states or not. If either  $\Lambda_{a-(k+3),b}^{(1)}$  or  $\Lambda_{a,b-(k+3)}^{(2)}$  is null, then so is  $V_{a,b}$ . This allows us to determine all physical states in a recursive way, exactly as in [2].

One easily sees that  $G_{1/2}^- \Lambda_{a,b-(k+3)}^{(2)} \equiv 0$  if  $b = k+2$  and  $a$  is arbitrary. Similarly,  $G_{1/2}^- \Lambda_{a-(k+3),b}^{(1)} \equiv 0$  if  $a = k+1$  and  $b$  is arbitrary. These lines of unphysical states are repeated for  $a$  and  $b$  modulo  $(k+3)$ , via the recurrence property just mentioned. The general scheme is displayed in Fig.1.

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<sup>†</sup> Note that these expressions are much simpler than the cumbersome expressions for the ground ring elements given in [1], where a different free-field realization was used.





**Fig.1:** Lines of null states cut through the domain  $0 \leq b \leq a$  of physical vertex operators,  $V_{a,b}$ . We indicated the *raison d'être* for each of the lines. Note that the lines are repeated modulo  $(k+3)$ . The bottom triangle represents the primary chiral ring of the matter model, and the other triangles represent the gravitational excitations.

There is one further line of unphysical states (also repeated modulo  $(k+3)$ ). This line appears because of the  $D$ -type screening in the  $N=2$  superconformal models. To see this, remember that the complete set of screeners, (9) plus (10), is obtained via Drinfeld-Sokolov reduction from the roots of the super-algebra  $SU(3|2)$ . In particular,  $R_1^D$  and  $R_2^D$  are associated with the bosonic  $SU(3)$  sub-algebra, while  $S_1^D$  is associated with the  $SU(2)$  sub-algebra. Indeed, a detailed analysis [19,15] shows that  $S_1^D$  contains the  $BRST$  current  $\exp(i\sqrt{\frac{2(k+2)}{k+3}}\varphi)$  corresponding to the  $SU(2)$  denominator factor in the minimal model part of the decomposition (14). Thus, this  $D$ -type screening constructs the minimal model Hilbert space using the denominator screening operator. There is, however, a subtlety, related to the presence of the nilpotent part  $N(z)$  of  $\hat{T}^\varphi(z)$  in (16). The fact that the quantum numbers of the numerator factor of  $\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$  in (14) are correlated with the quantum numbers of  $\frac{SU(3)_k}{SU(2)_{k+1} \times U(1)}$  means that the model has a special modular invariant. In particular, the total Hilbert space of the model can be mapped back onto itself by essentially trading of minimal model  $\varphi$ -momentum translations against translations in the momenta of the Coulomb gas realization of  $\frac{SU(3)_k}{SU(2)_{k+1} \times U(1)}$ . These  $\varphi$ -momentum translations are associated with the numerator factor of the minimal model  $\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$ . The fermionic screening is then precisely what further reduces the Hilbert space to a single copy of the physical states. These issues will be discussed more fully in [15].

The practical consequences of the foregoing in the present context is that because we have not performed the  $F$ -type screening, we must allow all values of the

$\varphi$ -momentum. However, because we have performed *BRST* reduction using the denominator screening charge as part of the *D*-type reduction, we have to take into account the corresponding null states. For general vertex operators,  $e^{ip\varphi}$ , this has two effects. First, one restricts to  $p \geq 0$ , or equivalently, one keeps only one state for each eigenstate of the zero mode of  $\widehat{T}^\varphi(z)$ . Secondly, for  $p = \frac{n}{\sqrt{2(k+2)(k+3)}}$ , one gets null states for  $n = (k+2)[1+2m(k+3)]$ ,  $m \in \mathbb{Z}$ . In terms of the  $N=2$  superconformal fields, one sees from (15) that  $\phi_2$  is orthogonal to  $\varphi$ , while  $\partial\varphi(z)\partial\phi_1(w) \sim -\frac{1}{\sqrt{2(k+2)}}\frac{1}{(z-w)^2}$ , hence  $V_{a,b}$  has  $\varphi$ -momentum equal to  $p = \frac{a-b}{\sqrt{2(k+2)(k+3)}}$ . The restriction  $p \geq 0$  means that we must take  $a \geq b$ , and the line of null states occurs at  $a-b = (k+2)$ . This means that the corresponding  $V_{a,b}$  are unphysical.

With Fig.1 we have thus recreated the picture of physical states found in [2]. However, our present discussion provides a more direct link to the structure of the underlying Lie algebra than the formulation in terms of Landau-Ginzburg fields. Specifically, we can associate to  $V_{a,b}$  a weight of  $SU(3)$  via:

$$\lambda_{a,b} = b\lambda_1 + (a-b)\lambda_2 + \rho ,$$

where  $\lambda_1, \lambda_2$  are the fundamental weights of  $SU(3)$ , and  $\rho = \lambda_1 + \lambda_2$  is the Weyl vector. Let now  $\alpha_1$  and  $\alpha_2$  be a system of simple roots. The requirement that  $\lambda_{a,b}$  lie in the interior of the the fundamental Weyl cone is:

$$\lambda_{a,b} \cdot \alpha_j > 0 , \quad j = 1, 2 , \quad \text{or} \quad a \geq b \geq 0 . \quad (20)$$

If we also require that

$$\lambda_{a,b} \cdot \alpha \not\equiv 0 \pmod{k+3} \quad (21)$$

for any root,  $\alpha$ , of  $SU(3)$ , we arrive at

$$\{ a-b \not\equiv k+2 , \quad b \not\equiv k+2 , \quad a \not\equiv k+1 \} \pmod{k+3} ,$$

which are precisely the physical state conditions that we evolved above.

The conditions (20) and (21) are indeed well-known in the representation theory of affine algebras, and just determine the highest weights of  $\widehat{SU}(3)$  (which are a subset of the highest weights of  $SU(3)$ ). Note that the states of the topological matter model (those with  $0 \leq b \leq a \leq k$ ) correspond to the subset of integrable highest weights, whereas their gravitational descendants correspond to non-integrable highest

weights, *i.e.*, to non-unitary representations. This is in accordance with our general considerations at the end of section 2.

The generalization of the foregoing to arbitrary  $N=2$   $W_n$  models is quite obvious, though it might be difficult to compute the details explicitly. The underlying Drinfeld-Sokolov reduction of  $SU(n|n-1)$  (with canonical gradation) leads to  $(2n-1)$   $D$ -type and to  $(2n)$   $F$ -type screeners, and the corresponding coset decomposes as

$$\frac{SU(n)_k \times SO(2n-2)}{SU(n-1)_{k+1} \times U(1)} \cong \frac{SU(n)_k}{SU(n-1)_k \times U(1)} \otimes \frac{SU(n-1)_k \times SU(n-1)_1}{SU(n-1)_{k+1}} \otimes U(1) . \quad (22)$$

Performing the  $D$ -type screening but not the  $F$ -type screening will give infinitely many copies of the “matter” Hilbert space involving translations of the momenta in the  $(n-1)$  directions of the  $SU(n)$  weight lattice. The  $D$ -type screening, along with positivity of the  $U(1)$  charge, will however restrict the momenta in the positive dual Weyl cone of  $SU(n)$ , and will introduce lines of null states associated with  $SU(n-1)$ . The other null states can then be identified by imposing equivariance. The result will almost certainly be that the physical states of topological  $W_n$ -matter coupled to topological  $W_n$ -gravity are generated by vertex operators associated with the ( $\rho$ -shifted)  $\widehat{SU}(n)$  highest weights,  $\lambda$ , which satisfy

$$\lambda \cdot \alpha > 0 , \quad \lambda \cdot \alpha \not\equiv 0 \bmod (k+n)$$

for all positive roots,  $\alpha$ , of  $SU(n)$ . We thus arrive at a very simple and satisfying algebraic characterization of the physical states, or ground rings, in these theories.

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